Itô's lemma and applications

Hideo NAGAI

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1 Itô's early works on stochastic differential equations

• Differential equations determining a Markoff processes, Journal of Pan-Japan mathematical colloquium No.1077(1942) (original Japanese: Zenkoku Sizyo Sugaku Danwakai-si)

• Stochastic differential equations in a differentiable manifold, Nagoya Mathematical Journal vol.1, 35-47(1950)

• On stochastic differential equations, Memoirs of American Mathematical Society vol. 4,1-51 (1951)

• On a formula concerning stochastic differentials, Nagoya math. Journal vol.3, 55-65 (1951)

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This is the first page of Ito's paper entitled "differential equations which determine Markov processes"

1077. Markoff過程ヲ定メル微分方程式

伊藤 清(内閣統計局)

ハシガキ

(I) 有限個ノ可能ナ場合 $a_1, a_2, \cdots a_m$ ヲ有シ，自然數ヲ徑數トスル simple Markoff process $x_1, x_2, \cdots$ ニ關シテ、多クノ遷移確率ヲ考ヘルコトが出來ル。例ヘバ $x_k = a_i$ ナル條件ノ下ニ於ケル $x_l = a_j$ ノ確率、或ハ $x_1 = a_{i_1}$, $x_2 = a_{i_2}, \cdots, x_n = a_{i_n}$ ナル條件ノ下ニ於ケル $x_{n+1} = a_{i_{n+1}}$ トナル確率等々。シカシ乍ラソレ等ハ結局 $x_k = a_i$ ノ時ノ $x_{k+1} = a_j$ ナル確率 $p_{ij}^{(k)}$ $(k = 1, 2, \cdots, i, j = 1, 2, \cdots m)$ ニ帰着セラレル。コレハ Kolmogoroff ノ本(*1) ニモ書イテアル。以後コレヲ基本的ナ遷移確率ト呼バウ。

更ニ可能ナ場合が有限デナクトモ、例ヘバ實數ヲ以テ標識ヅケラレル時ニハ、同ジコトガイヘルノハ云フマデモナイ。

併シナガラ徑數が自然數デナクテ實數ノ場合即チ continuous parameter ニ依存スル Markoff process ニ於テ、上ノコトハ如何ニナルカトイフコトハソレ程簡單デハナイ。(*2)

更ニ一般ニ可能ナ場合が實數ニヨリ標識付ケラレ、且ツ continuous parameter ニ依存スル simple Markoff

注意シテ)

$$P\left\{\sup_{0\leqq t\leqq 1}|y_m(\tau)|\geqq l\right\}\leqq\frac{1}{l^2}\sum_{\Delta_m}E(b^2_{t_{i-1}})(t_i-t_{i-1})$$

$m\to\infty$ トスレバ

$$P\left\{\sup_{0\leqq t\leqq 1}\left|\int_0^t b_\tau\,dx_\tau\right|\geqq l\right\}\leqq\frac{1}{l^2}\int_0^1 E(b^2_\tau)\,d\tau$$

III. 微分方程式ト積分方程式

§9. 本章デハ、微分方程式

(1) $dy_t=G(a(t,y_t),\ b(t,y_t))$

ヲ

(2) $y_0=C$

ナル初期條件ノ下ニ解クコトヲ目的トスル。茲ニ $G(\alpha,\beta)$ ハ平均値 α, 標準偏差 β ナル Gauss 分布ヲ表ハス。

先ヅ定理ヲ揭ゲル。

定理8.1. $0\leqq t\leqq 1$, $-\infty<y<\infty$ ナルトキ $a(t,y)$, $b(t,y)$ ガ何レモ t, y ニ關シテ連續デ、且ツ

(3) $|a(t,y)-a(t,y')|\leqq A|y-y'|$

$|b(t,y)-b(t,y')|\leqq B|y-y'|$

ヲ満スヤウナ常數 A, B ガ存在スルトスル。然ラバ x_t ヲ brownian motion トスルトキ

(4) $y_t=C+\int_0^t a(\tau,y_\tau)\,d\tau+\int_0^t b(\tau,y_\tau)\,dx_\tau$

ハ一ツ而シテ唯一ツノ解ヲ有シ、ソノ解ハ(1)ヲ満足スル。

§10. 上ノ積分方程式ノ解ノ存在証明(逐次近似法)

Lemma 10.1. $\{a_t\}$ ガ可動不連續点ナキ stochastic process デ $E(a_t^2)\leqq M(t)$ ナル連續函數 $M(t)$ ガ存在スルトスル。

而ラバ

(1) $$E\left\{\left(\int_t^s a_\tau\,d\tau\right)^2\right\}\leqq(s-t)\int_t^s M(\tau)\,d\tau$$

証明

(2) $$\left(\int_t^s a_\tau\,d\tau\right)^2\leqq(s-t)\int_t^s a_\tau^2\,d\tau$$

$$E\left\{\left(\int_t^s a_\tau\,d\tau\right)^2\right\}\leqq(s-t)E\left\{\int_t^s a_\tau^2\,d\tau\right\}\leqq(s-t)\int_t^s M(\tau)\,d\tau$$

最後ノ不等式ハ Lemma 5.1 ヲ用ヒテ証明スルコトガ出来ル。

備テセトニ於リ、$y_t^{(k)}$ $(k=1,2,\cdots\cdots)$ ヲ順々ニ次ノ關係式デ定義スル。

(3) $y_t^{(0)}=C$

(4) $y_t^{(k)}=C+\int_0^t a(\tau,y_\tau^{(k-1)})\,d\tau+\int_0^t b(\tau,y_\tau^{(k-1)})\,dx_\tau$

$(k=1,2,\cdots\cdots)$

先ヅ $y_t^{(k)}$ $(k=1,2,\cdots\cdots)$ ガ任意ノ定マッタ t $(0\leqq t\leqq 1)$

2 Chain rule for ODE

- ODE (Ordinary Differential Equation):

$$\frac{dX_t}{dt} = b(X_t)$$

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Namely,

$$df(X_t) = f'(X_t)b(X_t)dt$$

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Namely,

$$df(X_t) = f'(X_t)b(X_t)dt$$

$$\begin{aligned} f(X_T) - f(X_0) &= \int_0^T f'(X_t)b(X_t)dt \\ &= \lim_{|\Delta(T)|\to 0} \sum_i f'(X_{t_{k_i}})b(X_{t_{k_i}})(t_{i+1} - t_i) \end{aligned}$$

$$\Delta(T) := \{0 = t_0 < t_1, .. < t_N = T,\ N = 1, 2, ..\},\ t_i \le t_{k_i} \le t_{i+1}$$

3 Itô's differential rule

- SDE (Stochastic Differential Equation):

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

W_t :a standard Brownian motion

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$$df(X_t) = \{f'(X_t)b(X_t) + \frac{1}{2}f''(X_t)\sigma(X_t)^2\}dt + f'(X_t)\sigma(X_t)dW_t$$

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Why "$\frac{1}{2}f''(X_t)\sigma(X_t)^2$" comes in ?

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Why "$\frac{1}{2}f''(X_t)\sigma(X_t)^2$" comes in ?

- simplest case $(b \equiv 0,\ \sigma(x) = 1)$

(1) $$df(W_t) = \frac{1}{2}f''(W_t)dt + f'(W_t)dW_t$$

4 Brownian motion W_t

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- W_t is not of finite variation:

$$\sup_{\Delta(T)} \sum_{i=0}^{N} |W_{t_{i+1}} - W_{t_i}| = \infty, \quad \text{a.s.}$$

$$\Delta(T) := \{0 = t_0 < t_1 < t_2, \ldots, < t_N = T,\ N = 1, 2, \ldots\}$$

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- dW_t cannot be defined in an ordinary sense
- Then, how define dW_t ?
- Hint! look at the quadratic variation

$$\lim_{|\Delta(T)| \to 0} \sum |W_{t_{i+1}} - W_{t_i}|^2 = T$$

5 Brownian motion W_t (cont'd)

Note that for $0 = t_0 < t_1 \cdots < t_{N+1} = T$

$$W_T^2 - W_0^2 = \sum_{i=0}^{N} (W_{t_{i+1}}^2 - W_{t_i}^2)$$

$$= \sum_{i=0}^{N} (W_{t_{i+1}} - W_{t_i})^2 + \sum_{i=0}^{N} 2W_{t_i}(W_{t_{i+1}} - W_{t_i})$$

5 Brownian motion W_t (cont'd)

Note that for $0 = t_0 < t_1 \cdots < t_{N+1} = T$

$$W_T^2 - W_0^2 = \sum_{i=0}^{N} (W_{t_{i+1}}^2 - W_{t_i}^2)$$

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As $|\Delta(T)| \to 0$ we have

$$W_T^2 - W_0^2 = T + \int_0^T 2W_t dW_t \tag{5.1}$$

if we define

$$\int_0^T W_t dW_t := \lim_{|\Delta|(T) \to 0} \sum W_{t_i}(W_{t_{i+1}} - W_{t_i})$$

$$W_T^2 - W_0^2 = T + \int_0^T 2W_t dW_t \tag{5.1}$$

(5.1) means

$$f(W_T) - f(W_0) = \int_0^T \frac{1}{2} f''(W_t)dt + \int_0^T f'(W_t)dW_t \tag{5.2}$$

for $f(x) = x^2$ since $f'(x) = 2x, \quad f''(x) = 2$.

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for $f(x) = x^2$ since $f'(x) = 2x, \;\; f''(x) = 2$.

For general f (5.2) can be seen by using Taylor expansion

$$\begin{aligned} f(W_{t_{i+1}}) - f(W_{t_i}) &= f'(W_{t_i})(W_{t_{i+1}} - W_{t_i}) \\ &\quad + \tfrac{1}{2} f''(W_{t_i})(W_{t_{i+1}} - W_{t_i})^2 + o(|W_{t_{i+1}} - W_{t_i}|^2) \end{aligned}$$

6 Brownian motion W_t (cont'd)

$$\begin{aligned} \sum_i W_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) &= \sum_i (W_{t_{i+1}} - W_{t_i})(W_{t_{i+1}} - W_{t_i}) \\ &\quad + \sum_i W_{t_i}(W_{t_{i+1}} - W_{t_i}) \end{aligned}$$

implies

$$\lim_{|\Delta(T)|\to 0} \sum_i W_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) = T + \int_0^T W_t dW_t$$

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By defining

$$\int_0^T W_t \circ dW_t := \lim_{|\Delta(T)|\to 0} \sum_i \frac{W_{t_{i+1}} + W_{t_i}}{2}(W_{t_{i+1}} - W_{t_i})$$

we see that

$$\lim_{|\Delta(T)|\to 0} \sum_i W_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) = \int_0^T W_t \circ dW_t + \frac{1}{2}T$$

7 Definition of stochastic integral

Itô integral

For predictable $\varphi(t, \omega)$

$$\int_0^T \varphi(t, \omega) dW_t := \lim_{|\Delta(T)| \to 0} \sum_i \varphi(t_i, \omega)(W_{t_{i+1}} - W_{t_i})$$

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Stratonovich integral

For semimartingale $Y(t,\omega)$

$$\begin{aligned}
\int_0^T Y(t,\omega) \circ dW_t &:= \lim_{|\Delta(T)|\to 0} \sum_i \frac{Y(t_{i+1})+Y(t_i)}{2}(W_{t_{i+1}} - W_{t_i}) \\
&= \lim_{|\Delta(T)|\to 0} \sum_i \frac{2Y(t_i)+Y(t_{i+1})-Y(t_i)}{2}(W_{t_{i+1}} - W_{t_i}) \\
&= \int_0^T Y_t dW_t + \frac{1}{2}\langle Y, W\rangle_t
\end{aligned}$$

8 Itô's differential rule (II)

- Semi-martingale:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \equiv X_0 + A_t + M_t$$

8 Itô's differential rule (II)

- Semi-martingale:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \equiv X_0 + A_t + M_t$$

•Itô's differential rule for a semi-martingale:

$$\begin{aligned} f(X_t) - f(X_0) &= \textstyle\int_0^t \{f'(X_s)b(X_s) + \frac{1}{2}f''(X_s)\sigma(X_s)^2\}ds \\ &\quad + \textstyle\int_0^t f'(X_s)\sigma(X_s)dW_s \\ &= \textstyle\int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M, M\rangle_s \\ &= \textstyle\int_0^t f'(X_s) \circ dX_s \end{aligned}$$

$$\langle M, M\rangle_t = \int_0^t \sigma(X_s)^2 ds, \quad \langle f'(X_\cdot), X_\cdot\rangle_t = \int_0^t f''(X_s)\sigma(X_s)^2 ds$$

9 Stochastic exponential

Let

$$Y_t = e^{\int_0^t \sigma(X_s)dW_s} \equiv e^{M_t}, \quad f(x) = e^x$$

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Then, by Itô's formula we have

$$df(M_t) = dY_t = Y_t\sigma(X_t)dW_t + \frac{Y_t}{2}\sigma(X_t)^2 dt,$$

namely, $$\frac{dY_t}{Y_t} = \sigma(X_t)dW_t + \frac{1}{2}\sigma(X_t)^2 dt$$

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namely, $$\frac{dY_t}{Y_t} = \sigma(X_t)dW_t + \frac{1}{2}\sigma(X_t)^2dt$$

Thus we see that

$$Z_t = e^{\int_0^t \sigma(X_t)dW_t - \frac{1}{2}\int_0^t \sigma(X_t)^2dt}$$

satisfies

$$\frac{dZ_t}{Z_t} = \sigma(X_t)dW_t$$

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10 Kolmogorov equation

- SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \tag{10.1}$$

X_t^x: a solution to (10.1)

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X_t^x: a solution to (10.1)

- Itô's differential rule for $u = u(t,x)$

$$du(t, X_t^x) = \{\tfrac{\partial u}{\partial t}(t, X_t^x) + \tfrac{\partial u}{\partial x}(t, X_t^x)b(X_t^x) + \tfrac{1}{2}\tfrac{\partial^2 u}{\partial x^2}(t, X_t^x)\sigma(X_t^x)^2\}dt$$

$$+\tfrac{\partial u}{\partial x}(t, X_t^x)\sigma(X_t^x)dW_t$$

$$= \{\tfrac{\partial u}{\partial t}(t, X_t^x) + Lu(t, X_t)\}dt + \tfrac{\partial u}{\partial x}(t, X_t^x)\sigma(X_t^x)dW_t$$

where

$$Lu(t,x) := b(x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma(x)^2\frac{\partial^2 u}{\partial x^2}(t,x)$$

11 Kolmogorov equation (cont'd)

Set $u(s,x) := E[g(X^x_{T-s})]$. If it is smooth, then it satisfies

$$\begin{cases} \frac{\partial u}{\partial t}(s,x) + Lu(s,x) = 0, \quad 0 \le s < T \\ \\ u(T,x) = g(x) \end{cases}$$

11 Kolmogorov equation (cont'd)

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$$\begin{cases} \frac{\partial u}{\partial t}(s,x) + Lu(s,x) = 0, \quad 0 \le s < T \\ u(T,x) = g(x) \end{cases}$$

Because Itô's formula applied to $u(t,x)$ gives us

$$u(t, X^x_{t-s}) - u(s,x) = \int_0^{t-s} \{ (\tfrac{\partial u}{\partial t} + Lu)(s+r, X^x_r) \} dr$$

$$+ \int_0^{t-s} \tfrac{\partial u}{\partial x}(t, X^x_t) \sigma(X^x_t) dW_t$$

and Markov property implies

$$u(s,x) = E[g(X^x_{T-s})] = E[E[g(X^x_{T-s}) | X^x_{t-s}]]$$

$$= E[\,E[g(X^y_{T-t})\big|_{y=X^x_{t-s}} \, | X^x_{t-s}]] = E[u(t, X^x_{t-s})]$$

12 Optimal portfolio selection

Market model

Price of a riskless asset:

$$dS_t^0 = S_t^0 r(X_t)dt, \quad S_0^0 = 1$$

Price of a risky asset:

$$dS_t^1 = S_t^1\{\alpha(X_t)dt + \sum_{j=1}^{2} \sigma_j(X_t)dW_t^j\}, \quad S_0^1 = s_0$$

Economic factor:

$$dX_t = \beta(X_t)dt + \sum_{J=1}^{2} \lambda_j(X_t)dW_t^j, \quad X_0 = x$$

Wealth process

$$V_t = N_t^0 S_t^0 + N_t^1 S_t^1$$

$N_t^i,\ i = 0, 1$: numbers of share invested to i-th security S_t^i

Wealth process

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$N_t^i,\ i = 0, 1$: numbers of share invested to i-th security S_t^i

Portfolio proportion

$$h_t^i := \frac{N_t^i S_t^i}{V_t},\ i = 0, 1, \quad h_t^0 + h_t^1 = 1$$

Denote h_t^1 by h_t in what follows since $h_t^0 = 1 - h_t^1$.

Wealth process

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Denote h_t^1 by h_t in what follows since $h_t^0 = 1 - h_t^1$.

Dynamics of the wealth process

$$dV_t = V_t\{r(X_t) + h_t(\alpha(X_t) - r(X_t))\}dt + V_t h_t \sigma(X_t) dW_t,$$

$$V_0 = v_0$$

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$$V_0 = v_0$$

Since $\langle V \rangle_t = \int_0^t V_s^2 h_s^2 \sigma\sigma^*(X_s) ds$ we have by Itô's formula

$$\begin{aligned} dV_t^\gamma &= \gamma V_t^{\gamma-1} dV_t + \tfrac{1}{2}\gamma(\gamma-1)V_t^{\gamma-2} d\langle V \rangle_t \\ &= \gamma V_t^\gamma [\{r(X_t) + h_t(\alpha(X_t) - r(X_t))\}dt + h_t\sigma(X_t)dW_t] \\ &\quad + \tfrac{1}{2}\gamma(\gamma-1)V_t^\gamma h_t^2 \sigma\sigma^* dt \end{aligned}$$

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$$dV_t = V_t\{r(X_t) + h_t(\alpha(X_t) - r(X_t))\}dt + V_t h_t \sigma(X_t) dW_t,$$

$$V_0 = v_0$$

Since $\langle V \rangle_t = \int_0^t V_s^2 h_s^2 \sigma\sigma^*(X_s) ds$ we have by Itô's formula

$$\begin{aligned} dV_t^\gamma &= \gamma V_t^{\gamma-1} dV_t + \tfrac{1}{2}\gamma(\gamma-1)V_t^{\gamma-2} d\langle V \rangle_t \\ &= \gamma V_t^\gamma [\{r(X_t) + h_t(\alpha(X_t) - r(X_t))\}dt + h_t\sigma(X_t)dW_t] \\ &\quad + \tfrac{1}{2}\gamma(\gamma-1)V_t^\gamma h_t^2 \sigma\sigma^* dt \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{dV_t^\gamma}{V_t^\gamma} &= \gamma\{r(X_t) + h_t(\alpha(X_t) - r(X_t)) + \tfrac{1}{2}(\gamma-1)h_t^2\sigma\sigma^*(X_t)\}dt \\ &\quad + \gamma h_t \sigma(X_t) dW_t \end{aligned}$$

Once more again by Itô's formula we have

$$V_T^\gamma = v_0^\gamma e^{\gamma \int_0^T \eta(X_s, h_s) ds + \gamma \int_0^T h_s \sigma(X_s) dW_s - \frac{\gamma^2}{2} \int_0^T h_s^2 \sigma\sigma^*(X_s) ds}$$

where

$$\eta(x, h) = -\frac{1-\gamma}{2} h^2 \sigma\sigma^*(x) + h(\alpha(x) - r(x)) + r(x)$$

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Minimizing

$$\log E[V_T(h)^\gamma]$$

by selecting h amounts to minimizing

$$\log E[v_0^\gamma e^{\gamma \int_0^T \eta(X_s,h_s)ds + \gamma \int_0^T h_s\sigma(X_s)dW_s - \frac{\gamma^2}{2}\int_0^T h_s^2 \sigma\sigma^*(X_s)ds}]$$

Utility maximization and risk-sensitive portfolio optimization

Power utility maximization:

$$\sup_h \frac{1}{\gamma} E[V_T(h)^\gamma], \quad \gamma < 0$$

is equivalent to risk-senstive portfolio optimization:

$$\inf_h \log E[V_T(h)^\gamma], \quad \gamma < 0$$

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Through change of measure

$$\left.\frac{dP^h}{dP}\right|_{\mathcal{F}_T} = e^{\gamma \int_0^T h_s \sigma(X_s) dW_s - \frac{\gamma^2}{2} \int_0^T h_s^2 \sigma\sigma^*(X_s) ds},$$

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whose H-J-B (Hamilton-Jacobi-Bellman) equation is

(12.1)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\lambda\lambda^* \frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\lambda\lambda^* |\frac{\partial v}{\partial x}|^2 + \inf_h \{[\beta(x) + \gamma\lambda\sigma^*(x)h]\frac{\partial v}{\partial x} + \gamma\eta(x,h)\} = 0$$

$$v(T, x) = \gamma \log v_0$$

Analysis of the H-J-B equation (12.1) gives us optimal portfolio strategy

$$\hat{h}(t, X_t)$$

under suitable conditions, where

$$\hat{h}(t, x) = \frac{1}{1-\gamma}(\sigma\sigma^*(x))^{-1}(\alpha(x) - r(x)) + \sigma\lambda^*(x)\frac{\partial v}{\partial x}(t, x).$$

and $\frac{\partial v}{\partial x}(t, x)$ is the solution of the H-J-B equation.
Namely, $\hat{h}(t, X_t)$ attains the minimum of

$$\log E^h[v_0^\gamma e^{\gamma \int_0^T \eta(X_s, h_s) ds}], \quad \gamma < 0$$

which is equivalent to utility maximization.

Down-side risk minimization

Consider a down-side risk minimization on a long term:

$$J(\kappa) := \inf_h \liminf_{T\to\infty} \frac{1}{T} \log P(\frac{1}{T} \log V_T(h) \leq \kappa)$$

for a given target growth rate κ
and risk-sensitive portfolio optimization on a long term:

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We can show that

$$J(\kappa) = -\inf_{k\in(-\infty,\kappa]} \sup_{\gamma<0}\{\gamma k - \chi(\gamma)\} = -\sup_{\gamma<0}\{\gamma\kappa - \chi(\gamma)\}$$

for $\lim_{\gamma\to\infty} \chi'(\gamma) < \kappa < \chi'(0-)$ under suitable conditions.

Moreover, for given target growth rate κ we take $\gamma(\kappa)$ which attains

$$\sup_{\gamma<0}\{\gamma\kappa - \chi(\gamma)\} = \gamma(\kappa)\kappa - \chi(\gamma(\kappa))$$

and choose an optimal strategy $\hat{h}_t^{\gamma(\kappa)}$ which minimize

$$\liminf_{T\to\infty} \frac{1}{T} \log E[V_T(h)^\gamma].$$

Then, this $\hat{h}_t^{\gamma(\kappa)}$ attains the minimum of

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we can obtain $\hat{h}_t^{\gamma(\kappa)}$ through analysis of H-J-B equation of ergodic type corresponding to risk-sensitive portfolio optimization:

$$\chi(\gamma) := \inf_h \liminf_{T\to\infty} \frac{1}{T} \log E[V_T(h)^\gamma].$$

Thank you for your attention !